

ON KUMMER'S TWENTY-FOUR SOLUTIONS OF THE HYPERGEOMETRIC DIFFERENTIAL EQUATION

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ABSTRACT. The p -adic analyticity of the Boyarsky matrix associated with the hypergeometric function $F(a, b; c; x)$ has been investigated in an earlier article. The transformation of this matrix under translation of (a, b, c) by \mathbf{Z}^3 was determined at that time. This article gives the transformation of this matrix under the extended Kummer group. The p -adic implications of quadratic and higher-degree transformations remain open.

The object of this article is to explain the relation between Kummer's twenty-four solutions of this equation

$$(1) \quad \left(\lambda(1 - \lambda) \frac{d^2}{d\lambda^2} + (c - (a + b + 1)\lambda) \frac{d}{d\lambda} - ab \right) y = 0$$

and the p -adic theory of this equation.

For simplicity of exposition let us assume that a, b, c satisfy the condition

$$(2) \quad a, b, c - a, c - b \text{ all lie outside } \mathbf{Z}.$$

The unique solution of (1) analytic at $\lambda = 0$ is given by the classical formula

$$(3) \quad F(a, b, c; \lambda) = \sum_{s=0}^{\infty} \frac{(a)_s (b)_s}{(c)_s s!} \lambda^s,$$

where

$$(a)_s = \Gamma(a + s)/\Gamma(a), \quad s \in \mathbf{N}, s \geq 0.$$

The Euler integral formula

$$(4) \quad F(a, b, c; \lambda) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \omega_{a,b,c},$$

$$\omega_{a,b,c} = x^{b-1}(1-x)^{c-b-1}(1-x\lambda)^{-a} dx,$$

valid for $\operatorname{Re} c > \operatorname{Re} b > 0$ (a restriction which may be removed [7, p. 60, equation (13)] by changing the path of integration) shows that we are concerned with the periods of a certain differential. As is well known [5, 6; 11, p. 107; 12] the Kummer list may be obtained by transformation of this integral representation. Indeed there

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are twenty-four fractional linear transformations which map the set $\{0, 1, \infty\}$ into the set $\{0, 1, \infty, 1/\lambda\}$. Under such a substitution, $x = (\alpha t + \beta)/(\gamma t + \delta)$, the differential $\omega_{a,b,c}(x, \lambda)$ is transformed into a differential which (up to a factor independent of λ, t) is of the form

$$(5) \quad \lambda^{v_0}(1 - \lambda)^{v_1} \omega_{\hat{a}, \hat{b}, \hat{c}}(t, \hat{\lambda}),$$

where $\text{mod } \mathbf{Z}$, $\hat{a}, \hat{b}, \hat{c}$ are linear functions of (a, b, c) . We refer to the group of mappings $(a, b, c) \mapsto (\hat{a}, \hat{b}, \hat{c}) \text{ mod } \mathbf{Z}$ as the *Kummer group*. It is group theoretically isomorphic to the octahedral group. The mappings $\lambda \rightarrow \hat{\lambda}$ form a noncommutative group of order 6 well known as the group of fractional linear transformations which permute the points $\{0, 1, \infty\}$.

In this way, (cf. Table I) one deduces a solution

$$(6) \quad \lambda^{v_0}(1 - \lambda)^{v_1} F(\hat{a}, \hat{b}, \hat{c}, \hat{\lambda})$$

of (1) with constant factor adjusted so that we have six sets of solutions, each set consisting of four solutions which coincide at $\hat{\lambda} = 0$ and represent an eigenvector of the local monodromy at $\hat{\lambda} = 0$.

As noted by Riemann and re-examined by the more recent authors previously mentioned, the element $F(b, a, c; x)$ cannot be obtained in this way, i.e., the mapping $(a, b, c) \mapsto (b, a, c)$ does not lie in the Kummer group. By adjoining this element we obtain a group of 48 elements which we call the *extended Kummer group*. Since many authors (Kummer, Ince, Bateman project, Whittaker and Watson, Goursat, Poole) do not distinguish between these two groups and since the standard references do not give consistent lists from this point of view, we provide such a list in Table I as an introduction to the calculations summarized in Table II.

We have made available [3] a very lengthy treatment of the action of Frobenius upon the solutions of (1) subject to the further hypothesis

$$(7) \quad (a, b, c) \in \mathbf{Q} \cap \mathbf{Z}_p.$$

In particular the associated 2×2 matrix is (aside from possible poles at $0, \infty$) analytic in the region

$$(8) \quad |\lambda - 1| > |\pi| = |p|^{1/p-1}.$$

More recently [4], we constructed the associated Boyarsky matrix $\Gamma_B(a, b, c, \lambda)$ which we showed to be analytic as a function of (a, b, c) in each of the p^3 polydisks

$$(9) \quad \{(a, b, c) | |(a, b, c) + (\mu_1, \mu_2, \mu_3)| < |p|^{1-e_2}\},$$

where

$$e_2 = \frac{1}{(p-1)} \left(1 - \frac{1}{p} - \frac{1}{p-1} \right), \quad \mu_i \in \mathbf{N}, \quad 0 \leq \mu_i < p \quad (i = 1, 2, 3).$$

This analyticity of Γ_B was demonstrated subject to the condition

$$(10) \quad |\lambda| = 1 = |\lambda - 1|.$$

We showed that the contiguity relations of Gauss correspond to the transformation laws of Γ_B under translation of (a, b, c) by elements of \mathbf{Z}^3 . In the present work

we show that Kummer's list correspond to the transformation laws of $\Gamma_B(a, b, c; \lambda)$ under the twenty-four mappings $(a, b, c) \mapsto (\hat{a}, \hat{b}, \hat{c})$ listed in Table I. One of the applications is the analyticity of Γ_B as a function of (a, b, c, λ) with λ not restricted by (10). Another application (not developed here) is the possibility of eliminating the determinant calculation [4, equation (3.22.8)] from the proof [4, Theorem 3.22] of analyticity of Γ_B . Finally, we use duality [3, §4.7] to determine the relation between $\Gamma_B(a, b, c, \lambda)$ and $\Gamma_B(b, a, c, \lambda)$. Putting

$$(a, b, c) = -(\mu_1, \mu_2, \mu_3) + p(a', b', c'),$$

(a', b', c') bounded by $|p|^{-\epsilon_2}$, we find
(11)

$$\zeta(a, b, c) \begin{pmatrix} c-a, & 0 \\ 0, & c-b \end{pmatrix} \Gamma_B(a, b, c; \lambda) = \Gamma_B(b, a, c; \lambda) \begin{pmatrix} c'-a', & 0 \\ 0, & c'-b \end{pmatrix},$$

where $\zeta(a, b, c)$ is a scalar function of (a, b, c) independent of λ . Thus by a determinant calculation at $\lambda = 0$, ζ may be determined up to ± 1 . A precise determination is given in §6.

In this way we determine the transformation laws of Γ_B under the group extension of \mathbf{Z}^3 deduced from the 48 transformations generated by Table I and the transformation $(a, b, c) \mapsto (b, a, c)$.

We acknowledge the influence upon this article of the work [1] of Adolphson and Sperber on Bessel functions. We observe that subsection 3.2 is an ad hoc treatment in a special case of results known to Monsky and Washnitzer.

Notation. Ω is an algebraically closed extension of \mathbf{Q}_p complete under a valuation extending that of \mathbf{Q}_p ,

$$\pi p^{-1} = p.$$

$$D_1(a, r) = \{x \in \Omega \mid |x - a| < r\}.$$

$$f_{\vec{a}}(x, \lambda) = x^{a_2-1}(1-x)^{a_3-a_2}(1-x\lambda)^{-a_1}.$$

$$\omega_{\vec{a}, \lambda} = f_{\vec{a}}(x, \lambda) dx / (1-x).$$

$$X_{\vec{a}}(x, \lambda) = f_{\vec{a}}(x, \lambda) \left(\frac{1}{1-x} \right)^{-1} dx.$$

$$\underline{L}_{\lambda} = \Omega[x, 1/x, 1/(1-x), 1/(1-x\lambda)].$$

$$L_{\lambda} = \text{ring of analytic element on regions of the form}$$

$$|x| > \epsilon_0, \quad |1-x| > \epsilon, \quad |1-\lambda x| > \epsilon_{1/\lambda}, \quad |1/x| > \epsilon_{\infty}.$$

$$\text{ord}(a_1, a_2, a_3) = \inf_i \text{ord } a_i.$$

$$\text{ord } x = -\log|x|/\log p.$$

The $*$ of an invertible matrix denotes the transpose of the inverse.

$\gamma_p(a, b)$ is defined for $a, b \in \mathbf{Q} \cap \mathbf{Z}_p - \mathbf{Z}$, $pb - a \in \mathbf{Z}$ to be $\Gamma_p(a)\pi^{\mu}(-\pi)^{b-b_0}$, where $\mu \in \{0, 1, \dots, p-1\}$ $\mu \equiv pb - a \pmod{p}$, $pb_0 - a = \mu$ (cf. [3, equation (21.4.2); 4, equation (1.6)]).

$$e_2 = (1 - p^{-1} - (p-1)^{-1})(p-1)^{-1}.$$

1. Résumé of earlier work. We assume initially that

$$(1.1) \quad \begin{aligned} \vec{a} &= (a_1, a_2, a_3) \in (\mathbf{Q} \cap \mathbf{Z}_p)^3, \\ a_1, a_2, a_3 - a_1, a_3 - a_2 &\text{ all outside } \mathbf{Z}. \end{aligned}$$

The vector space $\omega_{\vec{a}} L_{\lambda} / d(f_{\vec{a}} L_{\lambda})$ is of dimension two with $\omega_{\vec{a}}$ and $(1-x)\omega_{\vec{a}}$ as a basis. We write

$$X_{\vec{a}}(x, \lambda) = \begin{pmatrix} (1-x)\omega_{\vec{a}} \\ \omega_{\vec{a}} \end{pmatrix}.$$

If then $\vec{c} \equiv \vec{a} \pmod{\mathbf{Z}}$, we may consider the change of basis matrix B defined by

$$(1.2) \quad \begin{pmatrix} (1-x)\omega_{\vec{c}} \\ \omega_{\vec{c}} \end{pmatrix} = X_{\vec{c}}(x, \lambda) = B(\vec{a}, \vec{c}, \lambda) X_{\vec{a}}(x, \lambda).$$

We have shown that B depends rationally upon \vec{a} and λ ; explicit formulae are given elsewhere [4, §3].

For

$$(1.3) \quad \lambda \neq 0, \infty, \quad |\lambda - 1| > |\pi|,$$

$$(1.4.1) \quad p\vec{b} - \vec{a} = \vec{\mu} \in \mathbf{Z}^3,$$

$$(1.4.2) \quad \vec{b} \in (Q \cap \mathbf{Z}_p)^3$$

we have defined an analytic lifting ψ_1 of the Cartier mapping which maps $\omega_{\vec{a}, \lambda} L_{\lambda}$ into $\omega_{\vec{b}, \lambda^p} L_{\lambda^p}$ by setting for $\xi \in L_{\lambda}$,

$$(1.5) \quad \psi_1 \left(\xi x f_{\vec{a}}(x, \lambda) \frac{dx}{x} \right) = \psi_0 \left(\frac{\xi f_{\vec{a}}(x, \lambda)}{f_{\vec{b}}(x^p, \lambda^p)} \right) x f_{\vec{b}}(x, \lambda^p) \frac{dx}{x},$$

where we identify $F_{\vec{a}, \vec{b}}(x, \lambda) = f_{\vec{a}}(x, \lambda) / f_{\vec{b}}(x^p, \lambda^p)$ with that element of L_{λ} which represents the indicated algebraic function of x and which, for a “generic” value of x , is approximated by

$$(1.5.1) \quad x^{p-1-\mu_2} (1-x)^{\mu_2-\mu_3} (1-\lambda x)^{\mu_1}$$

and where ψ_0 is the mapping of L_{λ} into L_{λ^p} defined by

$$(1.5.2) \quad (\psi_0 \xi)(x) = \sum_{y^p = x} \xi(y) / p y^{p-1}.$$

By passage to quotients we deduce from ψ_1 a mapping of $\omega_{\vec{a}, \lambda} L_{\lambda} / d(f_{\vec{a}}(x, \lambda) L_{\lambda})$ onto $\omega_{\vec{b}, \lambda^p} L_{\lambda^p} / d(f_{\vec{b}}(x, \lambda^p) L_{\lambda^p})$ whose matrix $\gamma(\vec{a}, \vec{b}; \lambda)$ we specify by writing

$$(1.6) \quad \psi_1 X_{\vec{a}}(x, \lambda) = \gamma(\vec{a}, \vec{b}; \lambda) X_{\vec{b}}(x, \lambda^p) \pmod{d(f_{\vec{b}}(x, \lambda^p) L_{\lambda^p})}.$$

With \vec{a}, \vec{b} satisfying (1.1), (1.4) we have shown [3]:

1.7. The matrix $\gamma(\vec{a}, \vec{b}; \lambda)$ is an analytic function of λ on the set (1.3), with at most poles at $\lambda = 0, \infty$.

The precise nature of this function at $\lambda = 0$ has been determined in [3, Chapters 24, 25].

We define $\Gamma_B(\vec{a}, \lambda)$ for \vec{a} satisfying (1.1) and λ satisfying (1.3) by writing

$$(1.8) \quad \Gamma_B(\vec{a}, \lambda) = \gamma(\vec{a}, \vec{b}, \lambda),$$

where we now insist that (1.4) holds with $\vec{\mu} = (\mu_1, \mu_2, \mu_3)$,

$$(1.8.1) \quad 0 \leq \mu_i \leq p-1, \quad i = 1, 2, 3.$$

We state the main result of our previous investigation [4, Theorem 3.22].

1.9. THEOREM. For

$$(1.9.1) \quad |\lambda| = 1 = |\lambda - 1|$$

Γ_B extends to an analytic function of \bar{a} for each fixed $\bar{\mu}$ satisfying (1.8.1) on the set

$$(1.9.2) \quad \text{ord } b_i > -e_2 \quad (i = 1, 2, 3).$$

1.10. Indeed, using the fact that the elements of

$$\underline{0} = \mathbf{Z}_p \left[\frac{1}{1-\lambda}, \frac{1}{1-\lambda^p}, \frac{1}{p} \left(\frac{1}{(\lambda-1)^p} - \frac{1}{\lambda^p-1} \right) \right]$$

are bounded by 1 for $|\lambda - 1| \geq 1$, the formulae used in the proof of 1.9 show that Γ_B extends to a function analytic in the pair (\bar{b}, λ) for \bar{b} restricted by (1.9.2) and λ restricted to the set

$$\{\lambda \mid |\lambda - 1| \geq 1, \lambda \neq \infty\}$$

with singularities at $\lambda = \infty$ being at worst pole of order μ_1 .

1.11. For $\bar{a} \in \Omega^3$ we define

$$(1.11.1) \quad G_{\bar{a}}(\lambda) = \begin{pmatrix} -\frac{a_3}{\lambda} & \frac{a_3 - a_1}{1 - \lambda} \\ \frac{a_3 - a_2}{\lambda} & \frac{a_1 + a_2 - a_3}{1 - \lambda} \end{pmatrix}.$$

We recall the equation

$$(1.11.2)_{\bar{a}} \quad \frac{d}{d\lambda}(u_1, u_2) = (u_1, u_2)G_{\bar{a}}(\lambda)$$

is satisfied by

$$(1.11.3) \quad (u_1, u_2) = ((a_3 - a_2)F(a_1, a_2, a_3 + 1; \lambda), a_3F(a_1, a_2, a_3; \lambda)).$$

For $z \neq 0, 1, \infty$, we define $C_{\bar{a}}(z, \lambda)$ to be the solution matrix of (1.11.2) $_{\bar{a}}$ which is normalized by the condition

$$(1.11.4) \quad C_{\bar{a}}(z, z) = I.$$

Thus $C_{\bar{a}}(z, \lambda)$ is an analytic function of λ for $(z - \lambda)$ sufficiently small. For \bar{a} satisfying (1.1), this matrix converges for

$$(1.11.5) \quad |\lambda - z| < |z| \text{Min}(1, |z - 1|).$$

We observe that if λ_0, λ both satisfy (1.11.5) then

$$(1.11.6) \quad C_{\bar{a}}(z, \lambda) = C_{\bar{a}}(z, \lambda_0)C_{\bar{a}}(\lambda_0, \lambda)$$

since both sides satisfy (1.11.2) $_{\bar{a}}$ and coincide at $\lambda = \lambda_0$. Putting $z = \lambda$ we conclude that

$$(1.11.7) \quad C_{\bar{a}}(\lambda_0, \lambda) = C_{\bar{a}}(\lambda, \lambda_0)^{-1}.$$

1.12. The deformation theory [3, Chapter 4, §6] may be interpreted as stating that if $\bar{a}, \bar{z}, \bar{\lambda}$ satisfy (1.1), (1.11.5), then

$$(1.12.1) \quad X_{\bar{a}}(x, \lambda) \equiv C_{\bar{a}}(z, \lambda)^t X_{\bar{a}}(x, z) \pmod{d(f_{\bar{a}}(x, z)L_z)}.$$

This is equivalent to the statement

$$(1.12.2) \quad \frac{\partial}{\partial \lambda} X_{\vec{a}}(x, \lambda) \equiv G_{\vec{a}}(\lambda)' X_{\vec{a}}(x, \lambda) \pmod{d(f_{\vec{a}}(x, \lambda) \underline{L}_{\lambda})}.$$

The relation between (1.11.2) and the matrix γ (equation (1.6)) may be written

$$(1.12.3) \quad C_{\vec{b}}(z^p, \lambda^p) \gamma(\vec{a}, \vec{b}, \lambda)' = \gamma(\vec{a}, \vec{b}, z)' C_{\vec{a}}(z, \lambda).$$

This may be deduced from (1.12.1) and (1.6).

1.13 Since one of our main tools is the deformation matrix $C_{\vec{a}}(z, \lambda)$ of subsection 1.11, we discuss the convergence without assuming validity of (1.1).

LEMMA. *The matrix $C_{\vec{a}}(z, \lambda)$ is analytic as a function of \vec{a}, z, λ subject to the condition*

$$(1.13.1) \quad \text{Inf}(0, \text{ord } \vec{a}) - \text{Sup}(0, \text{ord } z, \text{ord}(1 - z)) + \text{ord}(\lambda - z) > \frac{1}{p-1}.$$

PROOF. By a formal calculation, starting with (1.11.2), we find

$$(1.13.2) \quad \frac{d^s(u_1, u_2)}{d\lambda^s} = (u_1, u_2) G_{\vec{a}}^{(s)}(\lambda),$$

where the coefficients of $G_{\vec{a}}^{(s)}$ are polynomials over \mathbf{Z} in $\vec{a}, 1/\lambda, 1/(1 - \lambda)$ of degree bounded by s in each variable. By Taylor's theorem, for $z \neq 0, 1, \infty$

$$(1.13.3) \quad C_{\vec{a}}(z, \lambda) = \sum_{s=0}^{\infty} \frac{1}{s!} G_{\vec{a}}^{(s)}(z) \cdot (\lambda - z)^s.$$

The assertion follows immediately.

1.14. If x, λ, t are related (say rationally) then let $(\partial/\partial\lambda)x$ (resp. $(\partial/\partial\lambda)t$) designate differentiation with respect to keeping λ (resp. t) fixed. An elementary calculation shows that for any locally analytic function g of (x, λ) we have

$$\left(\frac{\partial}{\partial \lambda} \right)_t (g dx) = \left(\frac{\partial}{\partial \lambda} \right)_x (g dx) + d \left(g \cdot \left(\frac{\partial x}{\partial \lambda} \right)_t \right).$$

Thus for cohomological purposes we need not distinguish between the two possible definitions of $\partial/\partial\lambda$.

1.15. *Corrections of [4].*

In equation (3.14), $\hat{A}_{s,t}$ should read $\hat{A}_{s,\tau}$.

On p. 140, line 1*, T_1 should read $T_{1/\lambda}$.

In equation (3.15.1), $\lambda^{1+\mu_2-p}$ should read $\lambda^{1+\mu_2}$.

2. The Kummer transformations. In Table I we list 24 fractional linear transformations, T_j , which map the set $\{0, 1, \infty\}$ into the set $\{0, 1, \infty, 1/\lambda\}$. Clearly there will be one point, x_0 , not in the image. We define $1/\theta_m(\lambda)$ to be $T_m^{-1}(x_0)$, the inverse image under T_m of the missing point. The mappings $\{\theta_m\}$ form the six fractional linear transformations which permute the points $\{0, 1, \infty\}$. The twenty-four transformations $\{T_m\}$ do not form a group. The six elements $T_1, T_3, T_5, T_7, T_9, T_{11}$ do form a group, \mathcal{T} , and the full set consists of translates $T_j \mathcal{T}$, $j = 1, 2, 6, 10$, the classification being according to the four possible choices of excluded element x_0 .

In this section we will assume that $\vec{a}, \vec{b}, \lambda$ satisfy (1.1), (1.4) and (1.9.1).

Now let

$$(2.1) \quad x = T_m(t, \lambda), \quad 1 \leq m \leq 24.$$

We could write subject to (1.1)

$$(2.2) \quad X_{\vec{a}}(x, \lambda) \equiv H_m(\vec{a}, \lambda) X_{M_m(\vec{a})}(t, \theta_m(\lambda)) \pmod{d(f_{M_m(\vec{a})}(t, \theta_m(\lambda)) \underline{L}_{\theta_m(\lambda)}(t))}$$

with $M_m(\vec{a})$ unique modulo \mathbf{Z}^3 and H_m a 2×2 matrix independent of x and depending algebraically upon λ , with precise value depending upon the actual choice of $M_m(\vec{a})$. However, calculations show that we may choose $M_m(\vec{a})$ such that (2.2) may be replaced by an equality (valid incidentally for all \vec{a})

$$(2.3) \quad X_{\vec{a}}(x, \lambda) = H_m(\vec{a}, \lambda) X_{M_m(\vec{a})}(t, \theta_m(\lambda)),$$

and this construction fixes $M_m(\vec{a}) = (\hat{a}_1, \hat{a}_2, \hat{a}_3)$ uniquely. Table II displays the determination. It is useful to observe that the matrix $H_m(\vec{a}, \lambda)$ may itself be described as a product

$$(2.4) \quad H_m(\vec{a}, \lambda) = h_m(a, \lambda) N_m(\lambda)$$

of a scalar function, h_m depending upon \vec{a} and λ , with a 2×2 matrix function N_m depending rationally upon λ and independent of \vec{a} . The factors cannot be described uniquely by these conditions, and a choice is given in Table II for each m . We write for each m , following a formal calculation,

$$(2.5) \quad h_m(a, \lambda) = (-1)^{k(\vec{a})} \lambda^{k_0(\vec{a})} (1 - \lambda)^{k_1(\vec{a})},$$

where k, k_0, k_1 are linear (but not necessarily homogeneous) in \vec{a} and so subject to (1.1) we are dealing with an algebraic function of λ . We make no attempt to specify a branch of this function, but subject to (1.1) and (1.4), (letting σ denote the absolute Frobenius automorphism on the maximal unramified extension of \mathbf{Q}_p) we interpret the ratio

$$(2.6) \quad h_m(\vec{a}, \vec{b}, \lambda) \stackrel{\text{def}}{=} \frac{h_m(\vec{a}, \lambda)}{h_m^\sigma(\vec{b}, \lambda^p)} = (-1)^{k(\vec{a}) - pk(\vec{b})} \lambda^{k_0(\vec{a}) - pk_0(\vec{b})} \frac{(1 - \lambda)^{k_1(\vec{a})}}{(1 - \lambda^p)^{k_1(\vec{b})}}$$

which is well defined for λ not too close to 1 by the condition that

$$(1 - \lambda)^{k_1(\vec{a})} / (1 - \lambda^p)^{k_1(\vec{b})}$$

be approximated by $(1 - \lambda)^{k_1(\vec{a}) - pk_1(\vec{b})}$. A table of the linear functions k, k_0, k_1 may be deduced from Table II by observing the choice of h_m . In our application we shall be concerned with $h_m(\vec{a}, \vec{b}, \lambda)$ rather than with $h_m(\vec{a}, \lambda)$, and for this reason we have indicated $k(\vec{a})$ only mod \mathbf{Z} .

By differentiating (2.3) with respect to λ (cf. 1.14) and using equation (1.12.2) and letting $\lambda_m = \theta_m(\lambda)$ we find

$$(2.7) \quad G_{\vec{a}}(\lambda)' H_m(\vec{a}, \lambda) = \frac{d}{d\lambda} H_m(\vec{a}, \lambda) + \frac{d\theta_m}{d\lambda} H_m(\vec{a}, \lambda) (G_{M_m(\vec{a})}(\lambda_m))'.$$

TABLE I

List of fractional linear transformations $t \rightarrow T_m(t)$ mapping the set $\{0, 1, \infty\}$ to set $\{0, 1, \infty, 1/\lambda\}$. The symbol x_0 denotes the fourth element.

Solution of hypergeometric equation:

m	0	1	∞	x_0	$\theta_m(\lambda) = \hat{\lambda} = \frac{1}{T_m^{-1}(x_0)}$	$T_m(t)$	λ^{p_0}	$(1-\lambda)^{p_1}$	F	$(\hat{a}, \hat{b}, \hat{c}; \hat{\lambda})$	$\hat{\lambda}$ when $\lambda = 0$	exponent at: $\hat{\lambda} = 0$
1	0	1	∞	$1/\lambda$	λ	t	-	-	F	$(a, b, c; \lambda)$	0	0
2	1	0	$1/\lambda$	∞	λ	$\frac{1-t}{1-\lambda t}$	-	$(1-\lambda)^{c-a-b}$	F	$(c-a, c-b, c; \lambda)$	0	0
3	1	0	∞	$1/\lambda$	$\frac{\lambda}{\lambda-1}$	$1-t$	-	$(1-\lambda)^{-a}$	F	$(a, c-b, c; \frac{\lambda}{\lambda-1})$	0	0
4	0	1	$1/\lambda$	∞	$\frac{\lambda}{\lambda-1}$	$\frac{t}{1-\lambda+\lambda t}$	-	$(1-\lambda)^{-b}$	F	$(c-a, b, c; \frac{\lambda}{\lambda-1})$	0	0
5	0	∞	1	$1/\lambda$	$1-\lambda$	$\frac{t}{t-1}$	-	-	F	$(a, b, a+b+1-c; 1-\lambda)$	1	0
6	∞	0	$1/\lambda$	1	$1-\lambda$	$\frac{t-1}{\lambda t}$	λ^{1-c}	-	F	$(b+1-c, a+1-c, a+b+1-c; 1-\lambda)$	1	0
7	∞	0	1	$1/\lambda$	$1-\frac{1}{\lambda}$	$\frac{t-1}{t}$	λ^{-a}	-	F	$(a, a+1-c, a+b+1-c; 1-\frac{1}{\lambda})$	1	0
8	0	∞	$1/\lambda$	1	$1-\frac{1}{\lambda}$	$\frac{t}{\lambda(t-1)}$	λ^{-b}	-	F	$(b+1-c, b, a+b+1-c; 1-\frac{1}{\lambda})$	1	0
9	∞	1	0	$1/\lambda$	$1/\lambda$	$\frac{1}{t}$	$(-\lambda)^{-a}$	-	F	$(a, a+1-c, a+1-b; \frac{1}{\lambda})$	∞	a
10	1	∞	$1/\lambda$	0	$1/\lambda$	$\frac{1-t/\lambda}{1-t}$	$(-\lambda)^{b-c}$	$(1-\lambda)^{c-a-b}$	F	$(1-b, c-b, a+1-b; \frac{1}{\lambda})$	∞	a
11	1	∞	0	$1/\lambda$	$\frac{1}{1-\lambda}$	$\frac{1}{1-t}$	-	$(1-\lambda)^{-a}$	F	$(a, c-b, a+1-b; \frac{1}{1-\lambda})$	∞	a
12	∞	1	$1/\lambda$	0	$\frac{1}{1-\lambda}$	$\frac{t-1+\lambda}{\lambda t}$	$(-\lambda)^{1-c}$	$(1-\lambda)^{c-a-1}$	F	$(1-b, a+1-c, a+1-b; \frac{1}{1-\lambda})$	∞	a

Table I (continued)

m	0	1	∞	x_0	$\theta_m(\lambda) = \dot{\lambda} = \frac{1}{T_m^{-1}(x_0)}$	$T_m(t)$	λ^{x_0}	$(1-\lambda)^{r_1}$	F	$(\hat{a}, \hat{b}, \hat{c}; \hat{\lambda})$	$\hat{\lambda}$ when $\lambda=0$	exponent of $\hat{\lambda}$: $\lambda=0$
13	0	$1/\lambda$	∞	1	$1/\lambda$	t/λ	$(-\lambda)^{-b}$	-	F	$(b+1-c; b, b+1-a; \frac{1}{\lambda})$	∞	b
14	$1/\lambda$	0	1	∞	$1/\lambda$	$\frac{t-1}{t-\lambda}$	$(-\lambda)^{a-c}$	$(1-\lambda)^{c-a-b}$	F	$(c-a, 1-a, b+1-a; \frac{1}{\lambda})$	∞	b
15	0	$1/\lambda$	1	∞	$\frac{1}{1-\lambda}$	$\frac{t}{t+\lambda-1}$	-	$(1-\lambda)^{-b}$	F	$(c-a, b, b+1-a; \frac{1}{1-\lambda})$	∞	b
16	$1/\lambda$	0	∞	1	$\frac{1}{1-\lambda}$	$\frac{1-t}{\lambda}$	$(-\lambda)^{1-c}$	$(1-\lambda)^{c-b-1}$	F	$(b+1-c, 1-a, b+1-a; \frac{1}{1-\lambda})$	∞	b
17	∞	$1/\lambda$	0	1	λ	$\frac{1}{\lambda t}$	λ^{1-c}	-	F	$(b+1-c, a+1-c, 2-c; \lambda)$	0	$1-c$
18	$1/\lambda$	∞	1	0	λ	$\frac{t-(1/\lambda)}{t-1}$	λ^{1-c}	$(1-\lambda)^{c-a-b}$	F	$(1-b, 1-a, 2-c; \lambda)$	0	$1-c$
19	∞	$1/\lambda$	1	0	$\frac{\lambda}{\lambda-1}$	$1 + \left(\frac{1}{\lambda} - 1\right) \frac{1}{t}$	λ^{1-c}	$(1-\lambda)^{c-a-1}$	F	$(1-b, a+1-c, 2-c; \frac{\lambda}{\lambda-1})$	0	$1-c$
20	$1/\lambda$	∞	0	1	$\frac{\lambda}{\lambda-1}$	$\frac{1}{\lambda(1-t)}$	λ^{1-c}	$(1-\lambda)^{c-b-1}$	F	$(b+1-c, 1-a, 2-c; \frac{\lambda}{\lambda-1})$	0	$1-c$
21	1	$1/\lambda$	0	∞	$1-\lambda$	$\frac{1}{1+t(\lambda-1)}$	-	$(1-\lambda)^{c-a-b}$	F	$(c-a, c-b, c+1-a-b; 1-\lambda)$	1	$c-a-b$
22	$1/\lambda$	1	∞	0	$1-\lambda$	$\frac{1}{\lambda} + t\left(1-\frac{1}{\lambda}\right)$	λ^{1-c}	$(1-\lambda)^{c-a-b}$	F	$(1-b, 1-a, c+1-a-b, 1-\lambda)$	1	$c-a-b$
23	$1/\lambda$	1	0	∞	$1-\frac{1}{\lambda}$	$\frac{1}{(1-\lambda)t+\lambda}$	λ^{a-c}	$(1-\lambda)^{c-a-b}$	F	$(c-a, 1-a, c+1-a-b, 1-\frac{1}{\lambda})$	1	$c-a-b$
24	1	$1/\lambda$	∞	0	$1-\frac{1}{\lambda}$	$1 + \left(\frac{1}{\lambda} - 1\right)t$	λ^{b-c}	$(1-\lambda)^{c-a-b}$	F	$(1-b, c-b, c+1-a-b, 1-\frac{1}{\lambda})$	1	$c-a-b$

TABLE II

Transformation of $X_{a,b,c}(x, \lambda) = \begin{pmatrix} \omega_{a,b,c+1} \\ \omega_{a,b,c} \end{pmatrix}$ under $x = T_m(t, \lambda)$

$$\omega_{a,b,c} = x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx$$

$$X_{a,b,c}(x, \lambda) = \pm h_m(a, b, c; \lambda) N_m(\lambda) X_{\hat{a}, \hat{b}, \hat{c}}(t, \hat{\lambda})$$

m	T_m	$M_m(a, b, c) = (\hat{a}, \hat{b}, \hat{c})$				$\theta_m(\lambda) = \hat{\lambda}$		$h_m(a, b, c; \lambda)$	N_m
		\hat{a}	\hat{b}	\hat{c}	$\hat{\lambda}$	$\hat{\lambda}$	λ		
1	t	a	b	c		λ	λ	1	t
2	$\frac{1-t}{1-\lambda t}$	$1+c-a$	$c-b$	c		λ	λ	$(1-\lambda)^{1+c-a-b}$	$\begin{pmatrix} -1 & 1 \\ \lambda & 1-\lambda \end{pmatrix}$
3	$1-t$	a	$c-b$	c		$\frac{\lambda}{\lambda-1}$	$\frac{\lambda}{\lambda-1}$	$(1-\lambda)^{-a}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$
4	$\frac{t}{1-\lambda+\lambda t}$	$1+c-a$	b	c		$\frac{\lambda}{\lambda-1}$	$\frac{\lambda}{\lambda-1}$	$(1-\lambda)^{-b}$	$\begin{pmatrix} 1 & 0 \\ \lambda & 1-\lambda \end{pmatrix}$
5	$\frac{t}{t-1}$	a	b	$a+b-c$		$1-\lambda$	$1-\lambda$	$(-1)^b$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
6	$\frac{t-1}{\lambda t}$	$b+1-c$	$a-c$	$a+b-c$		$1-\lambda$	$1-\lambda$	$(-1)^b \lambda^{1-c}$	$\begin{pmatrix} 1 & 1 \\ 1-\lambda & 1 \end{pmatrix}$
7	$\frac{t-1}{t}$	a	$a-c$	$a+b-c$		$1-\frac{1}{\lambda}$	$1-\frac{1}{\lambda}$	$(-1)^b \lambda^{-a}$	$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$
8	$\frac{t}{\lambda(t-1)}$	$1+b-c$	b	$a+b-c$		$1-\frac{1}{\lambda}$	$1-\frac{1}{\lambda}$	$(-1)^b \lambda^{-b}$	$\begin{pmatrix} 1-(1/\lambda) & 1/\lambda \\ 1 & 0 \end{pmatrix}$
9	$1/t$	a	$a-c$	$a-b$		$1/\lambda$	$1/\lambda$	$(-1)^{c-a-b} \lambda^{-a}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$
10	$\frac{1-t/\lambda}{1-t}$	$1-b$	$c-b$	$a-b$		$1/\lambda$	$1/\lambda$	$\lambda^{b-c-1}(1-\lambda)^{c-a-b+1}$	$\begin{pmatrix} -1 & 1 \\ \lambda/(1-\lambda) & 0 \end{pmatrix}$

Table II (continued)

m	T_m	$M_m(a, b, c) = (\hat{a}, \hat{b}, \hat{c})$			$\theta_m(\lambda) = \hat{\lambda}$	$h_m(a, b, c; \lambda)$	N_m
		\hat{a}	\hat{b}	\hat{c}			
11	$1/(1-t)$	a	$c-b$	$a-b$	$1/(1-\lambda)$	$(-1)^{c-b}(1-\lambda)^{-a}$	$\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$
12	$\frac{t-1+\lambda}{\lambda t}$	$1-b$	$a-c$	$a-b$	$1/(1-\lambda)$	$(-1)^b \lambda^{-c}(1-\lambda)^{c-a}$	$\begin{pmatrix} 1 & 0 \\ -\lambda/(1-\lambda) & \lambda/(1-\lambda) \end{pmatrix}$
13	t/λ	$b+1-c$	b	$b+1-a$	$1/\lambda$	λ^{-b}	$\begin{pmatrix} 1/\lambda & 1-1/\lambda \\ 0 & 1 \end{pmatrix}$
14	$\frac{t-1}{t-\lambda}$	$1+c-a$	$1-a$	$b+1-a$	$1/\lambda$	$(-1)^{c-a-b} \lambda^{a-c-1}(1-\lambda)^{1+c-a-b}$	$\begin{pmatrix} 0 & 1 \\ -1/(1-\lambda) & 1 \end{pmatrix}$
15	$\frac{t}{t+\lambda-1}$	$1+c-a$	b	$1+b-a$	$1/(1-\lambda)$	$(-1)^b(1-\lambda)^{-b}$	$\begin{pmatrix} 0 & 1 \\ \lambda & 1-\lambda \end{pmatrix}$
16	$\frac{1-t}{\lambda}$	$1+b-c$	$1-a$	$1+b-a$	$1/(1-\lambda)$	$(-1)^{c-b} \lambda^{-c}(1-\lambda)^{c-b-1}$	$\begin{pmatrix} 1 & -\lambda \\ 0 & -\lambda \end{pmatrix}$
17	$1/\lambda t$	$1+b-c$	$a-c$	$1-c$	λ	$(-1)^{c-a-b} \lambda^{1-c}$	$\begin{pmatrix} 1 & (1/\lambda)-1 \\ 1 & -1 \end{pmatrix}$
18	$\frac{t-(1/\lambda)}{t-1}$	$1-b$	$1-a$	$1-c$	λ	$(-1)^{c-a-b} \lambda^{-c}(1-\lambda)^{1+c-a-b}$	$\begin{pmatrix} 0 & 1 \\ -\lambda/(1-\lambda) & 0 \end{pmatrix}$
19	$1 + \left(\frac{1}{\lambda} - 1\right) \frac{1}{t}$	$1-b$	$a-c$	$1-c$	$\lambda/(1-\lambda)$	$(-1)^{c-a-b} \lambda^{-c}(1-\lambda)^{c-a}$	$\begin{pmatrix} 0 & 1 \\ -\lambda & \lambda \end{pmatrix}$
20	$\frac{1}{\lambda(1-t)}$	$1+b-c$	$1-a$	$1-c$	$\lambda/(1-\lambda)$	$(-1)^{c-a-b} \lambda^{-c}(1-\lambda)^{c-b-1}$	$\begin{pmatrix} -\lambda & 1 \\ -\lambda & 0 \end{pmatrix}$
21	$\frac{1}{1-t(1-\lambda)}$	$1+c-a$	$c-b$	$1+c-a-b$	$1-\lambda$	$(-1)^{c-b}(1-\lambda)^{1+c-a-b}$	$\begin{pmatrix} -1 & 1 \\ -1 & \lambda/(\lambda-1) \end{pmatrix}$
22	$\frac{1}{\lambda} + t\left(1 - \frac{1}{\lambda}\right)$	$1-b$	$1-a$	$1+c-a-b$	$1-\lambda$	$(-1)^{c-b} \lambda^{-c}(1-\lambda)^{1+c-a-b}$	$\begin{pmatrix} 1 & 0 \\ 0 & \lambda/(\lambda-1) \end{pmatrix}$
23	$\frac{1}{(1-\lambda)t+\lambda}$	$1+c-a$	$1-a$	$1+c-a-b$	$1 - \frac{1}{\lambda}$	$(-1)^{c-b} \lambda^{a-c-1}(1-\lambda)^{1+c-a-b} \lambda^{a-c}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1/(1-\lambda) \end{pmatrix}$
24	$1 + \left(\frac{1}{\lambda} - 1\right)t$	$1-b$	$c-b$	$1+c-a-b$	$1 - \frac{1}{\lambda}$	$(-1)^{c-b} \lambda^{b-c-1}(1-\lambda)^{1+c-a-b}$	$\begin{pmatrix} -1 & 1 \\ 0 & 1/\lambda \end{pmatrix}$

This shows that the mapping

$$(2.8) \quad (v_1, v_2) \mapsto (u_1, u_2) = (v_1, v_2) H_m(\vec{a}, \lambda)^t$$

maps solutions of the equation

$$(1.11.2)_{M_m(\vec{a})} \quad \frac{d}{d\lambda_m}(v_1, v_2) = (v_1, v_2) G_{M_m(\vec{a})}(\lambda_m)$$

into solutions of (1.11.2) $_{\vec{a}}$. In particular for $z \neq 0, 1, \infty$

$$(2.9) \quad H_j(\vec{a}, z)^t C_{\vec{a}}(z, \lambda) = C_{M_j(\vec{a})}(\theta_j(z), \theta_j(\lambda)) H_j(\vec{a}, \lambda)^t,$$

the equation being interpreted by means of the fact that for λ close to z , the branch of the algebraic function H_j is specified by its determination at z .

This being said, the validity of (2.9) follows from the fact that each side represents a solution matrix of (1.11.2) $_{\vec{a}}$ and the sides coincide at $\lambda = z$.

We now define

$$(2.10) \quad K_m(\vec{a}, \vec{b}, \lambda) = \gamma(\vec{a}, \vec{b}, \lambda) * N_m(\lambda^p) * C_{M_m(\vec{b})}(\theta_m(\lambda^p), \theta_m(\lambda)^p) \\ \cdot \gamma(M_m(\vec{a}), M_m(\vec{b}), \theta_m(\lambda))^t N_m(\lambda)^t h_m(\vec{a}, \vec{b}, \lambda),$$

well defined for $\vec{a}, \vec{b}, \lambda$ satisfying (1.1), (1.4), (1.9.1). It may be continued analytically by means of Theorem 1.9 and subsection 1.13 so as to be well defined subject to (1.4.1), (1.9.1) provided

$$\text{ord } \vec{b} > \text{Sup} \left(-1 + \frac{1}{p-1}, -e_2 \right).$$

2.11. THEOREM. $K_m(\vec{a}, \vec{b}, \lambda)$ is a scalar function of (\vec{a}, \vec{b}) independent of λ provided μ_3 does not lie between μ_1 and μ_2 .

2.11.1. Note. In §3 we shall give a second analysis of K_m which removes all restrictions on $\vec{\mu}$.

PROOF. Let z, λ satisfy (1.9.1), $|\lambda - z|$ sufficiently small. Let a, b satisfy conditions (1.1) and (1.4). Then by (1.12.3)

$$(2.11.2) \quad C_{M_m(\vec{b})}(\theta_m(z)^p, \theta_m(\lambda)^p) \gamma(M_m(\vec{a}), M_m(\vec{b}), \theta_m(\lambda))^t \\ = \gamma(M_m(\vec{a}), M_m(\vec{b}), \theta_m(z))^t C_{M_m(\vec{a})}(\theta_m(z), \theta_m(\lambda)).$$

By the transitivity property (1.11.6) we have

$$(2.11.3) \quad C_{M_m(\vec{b})}(\theta_m(z)^p, \theta_m(\lambda)^p) = C_{M_m(\vec{b})}(\theta_m(z)^p, \theta_m(z^p)) C_{M_m(\vec{b})}(\theta_m(z^p), \theta_m(\lambda^p)) \\ \cdot C_{M_m(\vec{b})}(\theta_m(\lambda^p), \theta_m(\lambda)^p).$$

By (2.9) we have

$$(2.11.4) \quad C_{M_m(\vec{b})}(\theta_m(z^p), \theta_m(\lambda^p)) = H_m(\vec{b}, z^p)^t C_{\vec{b}}(z^p, \lambda^p) H_m(\vec{b}, \lambda^p)^*.$$

Substituting this expression in the right side of (2.11.3) and substituting the resulting formula in the left side of (2.11.2) and replacing $C_{M_m(\vec{a})}(\theta_m(z), \theta_m(\lambda))$ in the right side of (2.11.2) by the expression involving $C_{\vec{a}}(z, \lambda)$ deduced from (2.9), we obtain a lengthy equality which after multiplying on the left by

$$\left(C_{M_m(\vec{b})}(\theta_m(z)^p, \theta_m(z^p)) H_m(\vec{b}, z^p)^t \right)^{-1}$$

and on the right by $H_m(\vec{a}, \lambda)^t$ assumes the form

$$(2.11.5) \quad C_{\vec{b}}(z^p, \lambda^p) \gamma(\vec{a}, \vec{b}, \lambda)^t K_m(\vec{a}, \vec{b}, \lambda) = \gamma(\vec{a}, \vec{b}, z)^t K_m(\vec{a}, \vec{b}, z) C_{\vec{a}}(z, \lambda).$$

Multiplying each side of this equation on the left by the inverse of the corresponding side of (1.12.3) we deduce

$$(2.11.6) \quad K_m(\vec{a}, \vec{b}, \lambda) = C_{\vec{a}}(z, \lambda)^{-1} K_m(\vec{a}, \vec{b}, z) C_{\vec{a}}(z, \lambda).$$

This equation shows that the matrix function of the left side which is analytic (as a function of λ) on the set (1.9.1) is equivalent to a constant matrix, i.e. the eigenvalues of $K_m(\vec{a}, \vec{b}, \lambda)$ are independent of λ . If c is such an eigenvalue then there exists a solution (u_1, u_2) of (1.11.2) $_{\vec{a}}$ at z such that

$$(u_1, u_2) K_m(\vec{a}, \vec{b}, \lambda) = c(u_1, u_2).$$

If $K_m \neq cI_2$ then we may conclude that u_2/u_1 may be continued as a meromorphic function throughout the set (1.9.1). This would be in particular true if (u_1, u_2) were the unique bounded solution on $D(z, 1)$. This possibility of contradicting the existence of supersingular disks [3, Theorem 16.11] shows that $K_m(\vec{a}, \vec{b}, \lambda)$ is a scalar function whenever

$$(2.11.7) \quad \begin{cases} (p-1)(a_1, a_2, a_3) \in \mathbf{Z}_p, \\ \text{either } \mu_3 > \text{Max}(\mu_1, \mu_2) \text{ or } \mu_3 < \text{Min}(\mu_1, \mu_2). \end{cases}$$

Translating \vec{b} by \mathbf{Z}^3 does not affect these conditions and so using analyticity of K_m as a function of \vec{b} with $\vec{\mu}$ fixed permits us to deduce that $K_m(\vec{a}, \vec{b}, \lambda)$ is a scalar function of λ whenever μ_3 does not lie between μ_1 and μ_2 . It then follows from (2.11.6) that it is independent of λ .

REMARK. We do not believe that the quotient u_2/u_1 of components of a solution of (1.11.2) can be meromorphic on the set (1.9.1) except in the case in which u_2/u_1 is a rational function.

3. Variation of lifting of Frobenius. In this section we shall assume that $\vec{a}, \vec{b}, \lambda$ satisfy conditions (1.1), (1.4) and (1.9.1).

3.1. We observe that in the definitions (1.5) and (1.6) a choice has been made of $\lambda \rightarrow \lambda^p$ as a lifting of Frobenius for the parameter λ . We now generalize our formulation by considering a function $\lambda \rightarrow \phi(\lambda)$ analytic in a region of the type

$$(3.1.1) \quad \text{Sup}\{|\lambda|, |\lambda^{-1}|, |1 - \lambda|^{-1}\} < 1 + \varepsilon$$

for some $\varepsilon > 0$ and satisfying the further condition that

$$(3.1.2) \quad |\phi(\lambda) - \lambda^p| < 1$$

for all λ satisfying (1.9.1). We observe that by (1.12.1)

$$(3.1.3) \quad X_{\vec{b}}(x, \phi(\lambda)) = C_{\vec{b}}(\lambda^p, \phi(\lambda))^t X_{\vec{b}}(x, \lambda^p)$$

which reminds us that the space $\omega_{\vec{b}, \lambda^p} L_{\lambda^p}$ coincides with $\omega_{\vec{b}, \phi(\lambda)} L_{\lambda^p}$ and that $L_{\lambda^p} = L_{\phi(\lambda)}$. The mapping ψ_1 (equation (1.5)) may also be described in terms of ϕ , by means of the formula

$$(3.1.4) \quad \psi_1 \left(\xi x f_{\vec{a}}(x, \lambda) \frac{dx}{x} \right) = \psi_0(\xi F_{\vec{a}, \vec{b}}(x, \lambda, \phi(\lambda))) \cdot x f_{\vec{b}}(x, \phi(\lambda)) \frac{dx}{x},$$

where

$$(3.1.5) \quad F_{\vec{a}, \vec{b}}(x; \lambda, \phi(\lambda)) = \frac{f_{\vec{a}}(x, \lambda)}{f_{\vec{b}}(x^p, \phi(\lambda))}$$

which is to be understood again as that element of L_λ representing the indicated algebraic function of x which for generic x is approximated by (1.5.1). It will be convenient to explicitly denote the matrix of ψ_1 relative to this new basis of $\omega_{\vec{b}, \lambda^p} L_{\lambda^p} / d(f_{\vec{b}}(x, \lambda^p) L_{\lambda^p})$ by writing

$$(3.1.6) \quad \psi_1 X_{\vec{a}}(x, \lambda) = \gamma(\vec{a}, \vec{b}, \lambda, \phi(\lambda)) X_{\vec{b}}(x, \phi(\lambda)) \pmod{d(f_{\vec{b}}(x, \lambda^p) L_{\lambda^p})}.$$

In particular, $\gamma(\vec{a}, \vec{b}, \lambda) = \gamma(\vec{a}, \vec{b}, \lambda, \lambda^p)$ while in general

$$(3.1.7) \quad \gamma(\vec{a}, \vec{b}, \lambda, \phi(\lambda)) = \gamma(\vec{a}, \vec{b}, \lambda) C_{\vec{b}}(\phi(\lambda), \lambda^p)^t.$$

3.2. We now observe that in definitions (1.5) and (1.5.2) we have made a choice $x \mapsto x^p$ of lifting of Frobenius for the variable x . To generalize these definitions, let τ be an element of L_λ such that

$$(3.2.1) \quad |\tau(x) - x^p| \leq |p|$$

for generic x (and hence for all x such that

$$(3.2.2) \quad |x| = 1 = |x - 1| = |1 - x\lambda|).$$

We define

$$(3.2.3) \quad F_{\vec{a}, \vec{b}}(x, \tau(x); \lambda) = \frac{f_{\vec{a}}(x, \lambda)}{f_{\vec{b}}(\tau(x), \lambda^p)}$$

to be the unique element of L_λ which represents the indicated function obviously algebraic over L_λ and which for generic x is approximated by (1.5.1). We define a mapping $\psi_{\tau, 0}$ of L_λ into L_{λ^p} by

$$(3.2.4) \quad (\psi_{\tau, 0} \xi)(x) = \sum_{\tau(y)=x} \xi(y) / \tau'(y),$$

the sum being over the p values of y satisfying $\tau(y) = x$ such that y is approximately a p th root of x .

It follows from (3.2.1) that the approximate factorization of $\tau(y) - x$ represented by the congruence

$$\tau(y) - x \equiv y^p - x \pmod{p}$$

may be replaced by a precise factorization defined over L_{λ^p} [3, Chapter 20], i.e. there exists a unitary polynomial $\hat{\tau}(x, y)$ of degree p in y and with coefficients which are functions of x in L_{λ^p} such that for x satisfying $|x| = 1 = |x - 1| = |1 - x\lambda^p|$, the roots of the equation $\tau(y) = x$ allowed in the sum in the definition (3.2.4) are precisely the roots of the polynomial equation $\hat{\tau}(x, y) = 0$. Furthermore

$$(3.2.4.1) \quad \hat{\tau}(x, y) \equiv y^p - x \pmod{p}.$$

Thus in (3.2.4) we may write

$$(3.2.5) \quad (\psi_{\tau,0}\xi)(x) = \sum_{y|\hat{\tau}(x,y)=0} \xi(y)/\tau'(y).$$

We now define $\psi_{\tau,1}$, a mapping of $\omega_{\bar{a},\lambda}L_{\lambda}$ into $\omega_{\bar{b},\lambda^p}L_{\lambda^p}$, by setting for $\xi \in L_{\lambda^p}$,

$$\psi_{\tau,1}(\xi f_{\bar{a}}(x, \lambda) dx) = \psi_{\tau,0}(\xi F_{\bar{a},\bar{b}}(x, \tau(x); \lambda)) f_{\bar{b}}(x, \lambda^p) dx.$$

We show $\psi_{\tau,1}$ and ψ_1 are essentially the same on $\omega_{\bar{a},\lambda}L_{\lambda}$.

LEMMA 3.3. $\psi_{\tau,1} - \psi_1$ maps $\omega_{\bar{a},\lambda}$ into $d(L_{\lambda^p} f_{\bar{b}}(x, \lambda^p))$.

PROOF. Let $\xi \in L_{\lambda}$; define

$$\Delta = \psi_{\tau,0}\xi F_{\bar{a},\bar{b}}(x, \tau(x), \lambda) - \psi_0\xi F_{\bar{a},\bar{b}}(x, \lambda^p).$$

We assert that

$$(3.3.1) \quad \Delta \in \frac{1}{f_{\bar{b}}(x, \lambda^p)} \frac{d}{dx} L_{\lambda^p} f_{\bar{b}}(x, \lambda^p).$$

This would be enough to prove the assertion. We first make a local calculation, say with x in the generic disk,

$$(3.3.2) \quad \begin{cases} \hat{\tau}(x, y) = 0 & (\text{hence } \tau(y) = x), \\ x = \bar{y}^p \end{cases}$$

and letting $f_{\bar{a}}(\bar{y}, \lambda)$, $f_{\bar{b}}(x, \lambda^p)$ denote branches of $f_{\bar{a}}$ and $f_{\bar{b}}$ which are compatible in the sense that their ratio is $F_{\bar{a},\bar{b}}(\bar{y}, \lambda)$. We find

$$(3.3.3) \quad \Delta = \sum_y \xi(y) \frac{f_{\bar{a}}(y, \lambda)}{f_{\bar{b}}(x, \lambda^p)} \frac{1}{\tau'(y)} - \sum_{\bar{y}} \xi(\bar{y}) \frac{f_{\bar{a}}(\bar{y}, \lambda)}{f_{\bar{b}}(x, \lambda^p)} \frac{1}{p\bar{y}^{p-1}}$$

each of the sums being over solutions for y (resp. \bar{y}) of (3.3.2). By that equation

$$\frac{dy}{dx} = \frac{1}{\tau'(y)}, \quad \frac{d\bar{y}}{dx} = \frac{1}{p\bar{y}^{p-1}}$$

and so letting (y, \bar{y}) run over all pairs of solutions of (3.3.2) we obtain

$$(3.3.4) \quad p\Delta = \frac{1}{f_{\bar{b}}(x, \lambda^p)} \sum_{y, \bar{y}} \xi(y) f_{\bar{a}}(y, \lambda) \frac{dy}{dx} - \xi(\bar{y}) f_{\bar{a}}(\bar{y}, \lambda) \frac{d\bar{y}}{dx}.$$

Let us write $G(y) = \xi(y) f_{\bar{a}}(y, \lambda)$ which as a function of y is bounded on the generic disk. Hence

$$\left| \frac{1}{i!} \right| |G^{(i)}(y)| < |\xi|_{\text{gauss}}$$

and so we may compute

$$(3.3.5) \quad G(y) \frac{dy}{dx} - G(\bar{y}) \frac{d\bar{y}}{dx} = \frac{d}{dx} \sum_{i=0} G^{(i)}(\bar{y}) \rho^{i+1} / (i+1)!,$$

where

$$\rho = y - \bar{y}, \quad |\rho| < 1.$$

Using the definition of G , we find

$$(3.3.6) \quad \sum_{i=0} G^{(i)}(\bar{y}) \rho^{i+1} / (i+1) = \sum_{i=0} \frac{1}{i+1} \rho^{i+1} \sum_{l+k=i} \frac{1}{l!} \xi^{(l)}(\bar{y}) \frac{1}{k!} f_{\bar{a}}^{(k)}(\bar{y}, \lambda) \\ = f_{\bar{b}}(x, \lambda^p) \vec{F}_{\bar{a}, \bar{b}}(\bar{y}, \lambda) \eta(y, \bar{y}),$$

where

$$(3.3.7) \quad \eta(y, \bar{y}) = \sum_{i=0} \frac{(y - \bar{y})^{i+1}}{i+1} \sum_{l+k=i} \frac{1}{l!} \xi^{(l)}(\bar{y}) \frac{1}{k!} \frac{f_{\bar{a}}^{(k)}(\bar{y}, \lambda)}{f_{\bar{a}}(\bar{y}, \lambda)}.$$

For $g \in L_{\lambda}$ we may write uniquely

$$(3.3.8) \quad g(\bar{y}) = A_0(x) + A_1(x) \bar{y} + \cdots + A_{p-1}(x) \bar{y}^{p-1},$$

where each A_i lies in L_{λ^p} and $p|g|_{\text{gauss}} \geq |A_i|_{\text{gauss}}$. Indeed

$$pA_0(x) = \sum_{\bar{y}^p = x} g(\bar{y})$$

and similarly for the remaining A_i . This shows that

$$(3.3.9) \quad \eta(y, \bar{y}) = \sum_{i=0} \frac{(y - \bar{y})^{i+1}}{i+1} \sum_{j=0}^{p-1} A_{i,j}(x) \bar{y}^j,$$

where each $A_{i,j}$ lies in L_{λ} with uniformly bounded gauss norm. Using relation (3.2.4.1), we deduce

$$(y - \bar{y})^p = pB(x, y, \bar{y}),$$

where B is a polynomial in y, \bar{y} of degree bounded by $p-1$ in each variable with coefficients lying in L_{λ^p} and gauss norm bounded by unity. Since $p^{[m/p]}/m \rightarrow 0$ in \mathbb{Q}_p as $m \rightarrow \infty$, it follows that

$$\eta(y, \bar{y}) = \sum_{\substack{i < p \\ j < p}} \alpha_{i,j}(x) y^i \bar{y}^j,$$

where each $\alpha_{i,j}$ lies in L_{λ^p} . A similar result holds for the product $F_{\bar{a}, \bar{b}}(\bar{y}, \lambda) \eta(y, \bar{y})$.

By (3.3.4)

$$p\Delta = \frac{1}{f_b(x, \lambda^p)} \frac{d}{dx} (f_{\bar{b}}(x, \lambda^p) J(x)),$$

where $J(x) = \sum_{y, \bar{y}} F_{\bar{a}, \bar{b}}(\bar{y}, \lambda) \eta(y, \bar{y})$. It follows from the theory of symmetric function that $J \in L_{\lambda^p}$. This completes the proof of (3.3.1) and hence of the lemma.

4. Transformation of $\gamma(a, b, \lambda)$ under the Kummer group. The object of this section is to show that for $\vec{a}, \vec{b}, \lambda$ satisfying (1.1), (1.4) and (1.9) we have $K_m(\vec{a}, \vec{b}, \lambda) = I_2$. We make this explicit.

THEOREM. *Subject to the above hypotheses, for $1 \leq m \leq 24$ with h_m, N_m, M_m as given in Table II and (2.6),*

$$h_m(\vec{a}, \vec{b}, \lambda) \gamma(M_m(\vec{a}), M_m(\vec{b}); \theta_m(\lambda), \theta_m(\lambda^p)) = N_m(\lambda)^{-1} \gamma(\vec{a}, \vec{b}, \lambda) N_m(\lambda^p).$$

PROOF. We substitute $x = T_m(t, \lambda)$ in (1.6) and obtain from (2.3)

$$(4.1) \quad \begin{aligned} & \psi_{1,\tau} \left(H_m(\vec{a}, \lambda) X_{M_m(\vec{a})}(t, \theta_m(\lambda)) \right) \\ & \equiv \gamma(\vec{a}, \vec{b}, \lambda) H_m^a(\vec{b}, \lambda^p) X_{M_m(\vec{b})}(t, \theta_m(\lambda^p)) \pmod{\text{exact}}. \end{aligned}$$

The only point that needs clarification is the choice of branches of the transformed algebraic functions. We know that $f_{\vec{a}}(x, \lambda)$ and $f_{\vec{b}}(\tau(x), \lambda^p)$ must be interpreted as branches whose ratio is approximated by (1.5.1). The same must hold for their images under $T_m(t, \lambda)$. Thus for example for $m = 3$ computing formally ($t = 1 - x$) we obtain

$$(4.2) \quad \begin{aligned} f_{\vec{a}}(x, \lambda) &= (1 - t)^{a_2 - 1} (t)^{a_3 - a_2} (1 - \lambda(1 - t))^{-a_1} \\ &= (1 - \lambda)^{-a_1} t^{a_3 - a_2} (1 - t)^{a_2 - 1} (1 - \theta_3(\lambda)t)^{-a_1}. \end{aligned}$$

We may choose $\tau(x) = 1 - t^p$ and so we have formally

$$(4.3) \quad f_{\vec{b}}(\tau(x), \lambda^p) = (1 - \lambda^p)^{-b_1} t^{p(b_3 - b_2)} (1 - t^p)^{b_2 - 1} (1 - \theta_3(\lambda^p)t^p)^{-b_1}$$

and now we must interpret (4.3) and (4.2) so that (1.5.1) holds. This will indeed be the case if we do the following:

$$(4.4) \quad \text{interpret } (1 - \lambda)^{-a_1}, (1 - \lambda^p)^{-b_1} \text{ so that } \frac{(1 - \lambda)^{-a_1}}{(1 - \lambda^p)^{-b_1}} \equiv (1 - \lambda)^{\mu_1},$$

$$(4.5) \quad \text{interpret } f_{a_1, a_3 - a_2 + 1, a_3}(t, \theta_3(\lambda)) \text{ and } f_{b_1, b_3 - b_2 + 1, b_3}(t^p, \theta_3(\lambda^p))$$

so that the ratio is approximated by the rational function deduced from (1.5.1) with \vec{a}, \vec{b} replaced by $(a_1, a_3 - a_2 + 1, a_3)$, $(b_1, b_3 - b_2 + 1, b_3)$ and x, λ replaced by $(t, \theta_3(\lambda))$. In this way we deduce

$$(4.6) \quad \begin{aligned} & h_m(\vec{a}, \vec{b}; \lambda) N_m(\lambda) \psi_1(X_{M_m(\vec{a})}(t, \theta_m(\lambda))) \\ & = \gamma(\vec{a}, \vec{b}, \lambda) N_m(\lambda^p) X_{M_m(\vec{b})}(t, \theta_m(\lambda^p)) \pmod{\text{exact}}. \end{aligned}$$

By (3.1.6)

$$\begin{aligned} & \psi_1(X_{M_m(\vec{a})}(t, \theta_m(\lambda))) \\ & \equiv \gamma(M_m(\vec{a}), M_m(\vec{b}); \theta_m(\lambda), \theta_m(\lambda^p)) X_{M_m(\vec{b})}(t, \theta_m(\lambda^p)) \pmod{\text{exact}}. \end{aligned}$$

This completes the proof for $m = 3$. The other cases may be checked similarly. We observe, however, that strictly speaking it is not necessary to check each m . The group of M_m mod translations by \mathbf{Z}^3 is in fact the octahedral group [2, p. 408] and is generated by M_{13} and M_7 (M_{13} is of order two, M_7 is of order 3, M_{21} is of order 4

with a relation $M_{21}M_7M_{13} = I_3$). Since $M_9M_3 = M_7$, the group is generated by M_3 , M_9 , M_{13} . The transformations T_9 and T_{13} are relatively simple to handle and so the transformation law for M_3 is the most interesting. This completes our treatment of the theorem.

5. Twist transformation. We now find the relation between $\gamma(\vec{a}, \vec{b}, \lambda)$ and $\gamma(\vec{a}'^w, \vec{b}'^w, \lambda)$, where $\vec{a}'^w = (a_2, a_1, a_3)$. As usual we shall assume conditions (1.1), (1.4) and (1.9.1) to be satisfied.

THEOREM.

(5.0)

$$\gamma(\vec{a}'^w, \vec{b}'^w, \lambda)\zeta(\vec{a}, \vec{b}) = \begin{pmatrix} a_3 - a_1, & 0 \\ 0, & a_3 - a_2 \end{pmatrix} \gamma(\vec{a}, \vec{b}, \lambda) \begin{pmatrix} b_3 - b_1, & 0 \\ 0, & b_3 - b_2 \end{pmatrix}^{-1},$$

where

$$\zeta(\vec{a}, \vec{b}) = p \frac{\gamma_p(1 + a_3 - a_2, 1 + b_3 - b_2)}{\gamma_p(1 - a_2, 1 - b_2)\gamma_p(a_1, b_1)\gamma_p(a_3 - a_1, b_3 - b_1)(-1)^{\mu_2}}$$

is a scalar function of \vec{a}, \vec{b} , independent of λ .

PROOF. By duality [4, (3.22.7)]

$$\begin{aligned} (5.1) \quad \gamma(\vec{a}, \vec{b}, \gamma)^t & \begin{pmatrix} (a_3 - a_1)\lambda, & 0 \\ 0, & (a_3 - a_2)(1 - \lambda) \end{pmatrix} \gamma(\kappa(\vec{a}), \kappa(\vec{b}), \lambda) \\ & = p \begin{pmatrix} (b_3 - b_1)\lambda^p, & 0 \\ 0, & (b_3 - b_2)(1 - \lambda^p) \end{pmatrix}, \end{aligned}$$

where $\kappa(\vec{a}) = (1 - a_1, 1 - a_2, 1 - a_3)$. We observe that $M_{18}(\vec{a}'^w) = \kappa(\vec{a})$, while $h_{18}(\vec{a}, \vec{b}, \lambda)$ is unchanged under $(\vec{a}, \vec{b}) \rightarrow (\vec{a}'^w, \vec{b}'^w)$. Thus

(5.2)

$$\begin{pmatrix} 0, & 1 - \lambda \\ -\lambda, & 0 \end{pmatrix} \gamma(\kappa(\vec{a}), \kappa(\vec{b}), \lambda) = \gamma(\vec{a}'^w, \vec{b}'^w, \lambda) \begin{pmatrix} 0, & 1 - \lambda^p \\ -\lambda^p, & 0 \end{pmatrix} \frac{1}{\hat{h}_{18}(\vec{a}, \vec{b}, \lambda)},$$

where $\hat{h}_{18}(\vec{a}, \vec{b}, \lambda) = h_{18}(\vec{a}, \vec{b}, \lambda)(1 - \lambda^p)/(1 - \lambda)$. Substituting in this equation gives

$$\begin{aligned} (5.3) \quad \gamma(\vec{a}, \vec{b}, \lambda)^t & \begin{pmatrix} 0, & -(a_3 - a_1) \\ a_3 - a_2, & 0 \end{pmatrix} \gamma(\vec{a}'^w, \vec{b}'^w, \lambda) \\ & = p \hat{h}_{18}(\vec{a}, \vec{b}, \lambda) \begin{pmatrix} 0, & -(b_3 - b_1) \\ b_3 - b_2, & 0 \end{pmatrix}. \end{aligned}$$

We multiply the right by $\gamma(\vec{a}'^w, \vec{b}'^w, \lambda)^{-1}$, take the transpose, multiply on the left by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and deduce

$$(5.4) \quad \text{right side of (5.0)} = p \hat{h}_{18}(\vec{a}, \vec{b}, \lambda) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \gamma(\vec{a}'^w, \vec{b}'^w, \lambda)^* \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If X is a 2×2 matrix then

$$(5.5) \quad \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} \det X = X \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix} X^t$$

and so the right side of (5.4) is

$$(5.6) \quad p\hat{h}_{18}(\vec{a}, \vec{b}, \lambda) \gamma(\vec{a}^{tw}, \vec{b}^{tw}, \lambda) \cdot \frac{1}{\det \gamma(\vec{a}^{tw}, \vec{b}^{tw}, \lambda)}.$$

Thus (5.0) is valid if ζ is the scalar function

$$(5.7) \quad \zeta = p\hat{h}_{18}(\vec{a}, \vec{b}, \lambda) / \det \gamma(\vec{a}^{tw}, \vec{b}^{tw}, \lambda).$$

Now by means of [3, Theorem 25]

$$(5.8) \quad \det \gamma(\vec{a}^{tw}, \vec{b}^{tw}, \lambda) = \lambda^{\mu_3} \frac{(1 - \lambda^p)^{b_1 + b_2 - b_3}}{(1 - \lambda)^{a_1 + a_2 - a_3}} (-1)^{\mu_1 - \mu_3} \\ \times \frac{\gamma_p(1 - a_2, 1 - b_2) \gamma_p(a_1, b_1) \gamma_p(a_3 - a_1, b_3 - b_1)}{\gamma_p(1 + a_3 - a_2, 1 + b_3 - b_2)}$$

On the other hand, by (2.6)

$$(5.9) \quad \hat{h}_{18}(\vec{a}, \vec{b}, \lambda) = (-1)^{\mu_3 - \mu_1 - \mu_2} \lambda^{\mu_3} \frac{(1 - \lambda)^{a_3 - a_1 - a_2}}{(1 - \lambda^p)^{b_3 - b_1 - b_2}}.$$

This completes the proof of the theorem.

5.10. We now give an independent treatment of the theorem based purely on the differential equation point of view. We shall assume $a_1 \notin \mathbf{Z}$. This will put us in the nonlogarithmic case for the solutions of zero. There is a similar and indeed easier treatment of the logarithmic case. Let us write (cf. [3, Lemma 24.1])

$$(5.10.1) \quad V_{\vec{a}}(\lambda) = \begin{pmatrix} (a_3 - a_2)F(a_1, a_2, 1 + a_3; \lambda) \\ (1 - a_3)F(a_1 - a_3, a_2 - a_3, 1 - a_3; \lambda) \\ a_3 F(a_1, a_2, a_3; \lambda) \\ (a_3 - a_1)\lambda F(a_1 + 1 - a_3, a_2 + 1 - a_3, 2 - a_3; \lambda) \end{pmatrix}.$$

Then by [3, Lemma 24.3]

$$(5.10.2) \quad V_{\vec{b}}(\lambda^p) \gamma(\vec{a}, \vec{b}, \lambda)^t = \Delta(\vec{a}, \vec{b}, \lambda) V_{\vec{a}}(\lambda),$$

$$(5.10.3) \quad V_{\vec{b}^{tw}}(\lambda^p) \gamma(\vec{a}^{tw}, \vec{b}^{tw}, \lambda)^t = \Delta(\vec{a}^{tw}, \vec{b}^{tw}, \lambda) V_{\vec{a}^{tw}}(\lambda),$$

where $\Delta(\vec{a}, \vec{b}, \lambda)$ is a diagonal matrix given explicitly in this nonlogarithmic case [3, Theorem 25] by

$$(5.10.4) \quad \Delta(\vec{a}, \vec{b}, \lambda) = \begin{pmatrix} \theta_1(\vec{a}, \vec{b}) & 0 \\ 0 & \lambda^{\mu_3} \theta_4(\vec{a}, \vec{b}) \end{pmatrix},$$

where

$$(5.10.5) \quad \begin{aligned} \theta_1(a, b) &= \frac{\gamma_p(a_2, b_2) \gamma_p(a_3 - a_2, b_3 - b_2)}{\gamma_p(1 + a_3, 1 + b_3)}, \\ \theta_4(\vec{a}, \vec{b}) &= (-1)^{\mu_2 - \mu_3} \frac{\gamma_p(a_3 - 1, b_3 - 1) \gamma_p(1 - a_1, 1 - b_1)}{\gamma_p(1 + a_3 - a_1, 1 + b_3 - b_1)}. \end{aligned}$$

It is easily deduced from (5.10.1) that

$$(5.10.6) \quad V_{\vec{a}^{tw}} = \begin{pmatrix} a_3 - a_2 & 0 \\ 0 & a_3 - a_1 \end{pmatrix}^{-1} V_{\vec{a}} \begin{pmatrix} a_3 - a_1 & 0 \\ 0 & a_3 - a_2 \end{pmatrix}$$

and similarly for $V_{\vec{b}^{tw}}$ in terms of $V_{\vec{b}}$. Substituting these in (5.10.3) we find

$$(5.10.7) \quad \begin{aligned} & \begin{pmatrix} b_3 - b_1 & 0 \\ 0 & b_3 - b_2 \end{pmatrix} \gamma(\vec{a}^{tw}, \vec{b}^{tw}, \lambda)^t \begin{pmatrix} a_3 - a_1 & \\ & a_3 - a_2 \end{pmatrix}^{-1} \\ &= V_{\vec{b}}(\lambda^p)^{-1} \begin{pmatrix} b_3 - b_2 & 0 \\ 0 & b_3 - b_1 \end{pmatrix} \Delta(\vec{a}^{tw}, \vec{b}^{tw}, \lambda) \begin{pmatrix} a_3 - a_2 & \\ & a_3 - a_1 \end{pmatrix}^{-1} V_{\vec{a}}(\lambda). \end{aligned}$$

From this the proof of the theorem is reduced to verifying

$$(5.10.8) \quad \zeta(\vec{a}, \vec{b}) \begin{pmatrix} b_3 - b_2 & 0 \\ 0 & b_3 - b_1 \end{pmatrix} \Delta(\vec{a}^{tw}, \vec{b}^{tw}, \lambda) \begin{pmatrix} a_3 - a_2 & 0 \\ 0 & a_3 - a_1 \end{pmatrix}^{-1} = \Delta(\vec{a}, \vec{b}, \lambda).$$

This verification follows easily from the explicit formulae (5.10.4) and (5.10.5).

In the logarithmic case ($a_3 = b_3 = 1$) we put

$$(5.10.1)' \quad V_{\vec{a}}(\lambda) = \begin{pmatrix} (1 - a_2)F(a_1, a_2, 2, \lambda) & F(a_1, a_2, 1, \lambda) \\ (1 - a_2)H(a_1, a_2, 2, \lambda) & H(a_1, a_2, 1, \lambda) \end{pmatrix},$$

where for $n \in \mathbb{N}$, $n \geq 1$, $H(a_1, a_2, n, \lambda)$ is defined by the conditions that it lie in $\mathbb{Q}(a_1, a_2)((\lambda))$ and that

$$y = H(a_1, a_2, n, \lambda) + F(a_1, a_2, n, \lambda) \log \lambda$$

be a formal solution of $(\lambda(\delta + a_1)(\delta + a_2) - \delta(\delta + n - 1))y = 0$, where $\delta = \lambda d/d\lambda$.

Equation (5.10.2) now takes the form

$$(5.10.2)' \quad V_{\vec{b}}(\lambda^p) \gamma(\vec{a}, \vec{b}, \lambda)^t = \theta \begin{pmatrix} 1 & 0 \\ \nu & p \end{pmatrix} V_{\vec{a}}(\lambda),$$

where [3, Theorem 2.6.1]

$$(5.10.4)' \quad \theta = (-1)^{\mu_2}, \quad \frac{\nu}{p} = \frac{\Gamma'_p}{\Gamma_p}(a_1) + \frac{\Gamma'_p}{\Gamma_p}(a_2) - 2 \frac{\Gamma'_p}{\Gamma_p}(1).$$

The second verification of the theorem in this case causes no difficulty.

6. Hypergeometric-Boyarsky gamma function. Up to this point we have studied the transformation laws of $\gamma(\vec{a}, \vec{b}; \lambda)$ subject to conditions (1.1), (1.4) and (1.9.1).

We now recall the hypergeometric Boyarsky gamma function. Let

$$(6.1) \quad \begin{aligned} S' &= \{ \vec{z} = (z_1, z_2, z_3) \mid \text{ord } z_i > -e_2, i = 1, 2, 3 \}, \\ S &= \{ \vec{x} = -\vec{\mu} + p\vec{z} \mid z \in S', 0 \leq \mu_1 < p, \mu_i \in \mathbb{N}, i = 1, 2, 3 \}. \end{aligned}$$

We define [4, §3.19, Theorem 3.22] $\Gamma_B(\vec{x}, \lambda)$ for $\vec{x} \in S$ and λ satisfying (1.9.1) by the condition that it is an analytic function of \vec{x} (for $\vec{\mu}$ fixed) and that

$$(6.2) \quad \Gamma_B(\vec{a}, \lambda) = \gamma(\vec{a}, \vec{b}, \lambda)$$

for all \vec{a} satisfying (1.1) with \vec{b} chosen uniquely so that (1.4) holds with

$$(6.3) \quad \mu_1, \mu_2, \mu_3 \text{ in } [0, p-1].$$

We also recall [4, (3.2)] the change of basis formula for \vec{a}, \vec{b} satisfying (1.1), (1.4), $m_1, m_2, m_3, n_1, n_2, n_3$ all in \mathbb{Z} .

$$(6.4) \quad \begin{aligned} &\gamma(\vec{a} + (m_1, m_2, m_3), \vec{b} + (n_1, n_2, n_3), \lambda) \\ &= B(\vec{a}, \vec{a} + (m_1, m_2, m_3), \lambda) \gamma(\vec{a}, \vec{b}; \lambda) B(\vec{b}, \vec{b} + (n_1, n_2, n_3), \lambda^p)^{-1}, \end{aligned}$$

where the matrix B is defined by (1.2).

THEOREM. *In the notation of Table II, with $\vec{\mu}$ satisfying (6.3) let*

$$(6.5) \quad \hat{\mu}_m = -M_m(0) + \frac{1}{p} [M_m(-\vec{\mu}) + \text{Rep}(-M_m(-\vec{\mu}))],$$

where Rep denotes the mapping of \mathbb{Z}_p into $\{0, 1, \dots, p-1\}$ obtained by taking a representative of class modulo p . For λ satisfying (1.9.1) and $\vec{x} = -\mu + p\vec{z}$ lying in S ($\vec{z} \in S'$) we have

$$(6.6) \quad \begin{aligned} &h_m(\vec{x}, \vec{z}, \lambda) \Gamma_B(M_m(\vec{x}), \theta_m(\lambda)) \\ &= N_m(\lambda)^{-1} \Gamma_B(\vec{x}, \lambda) N_m(\lambda^p) C_{M_m(\vec{z})}(\theta_m(\lambda)^p, \theta_m(\lambda^p))^t \\ &\quad \times B(M_m(\vec{z}), M_m(\vec{z}) + \hat{\mu}_m, \theta_m(\lambda)^p)^{-1}. \end{aligned}$$

PROOF. We recall that $h_m(\vec{a}, \vec{b}; \lambda)$ is a product of the form

$$\lambda^{\mu_1} \times (1 - \lambda)^{a_1} / (1 - \lambda^p)^{b_1}.$$

The second factor is of the form

$$(1 - \lambda)^{\mu_1} ((1 - \lambda)^p / (1 - \lambda^p))^{b_1}$$

and so we conclude that $h_m(\vec{x}, \vec{z}, \lambda)$ is analytic, as a function of \vec{x}, λ subject to the condition

$$(6.7) \quad \text{Inf}(0, \text{ord } \vec{z}) + 1 > \frac{1}{p-1} + \text{Sup}(0, (p-1) \text{ord}(\lambda - 1)),$$

and hence is analytic subject to (1.9.1) for \vec{x} in S . Again subject to (1.9.1), $\theta_m(\lambda)$ cannot lie in $D(1, 1^-)$ and so by Lemma 1.13 the matrix $C_{M_m(\vec{z})}(\theta_m(\lambda)^p, \theta_m(\lambda^p))$ is analytic for

$$(6.8) \quad 1 + \text{ord } \vec{z} > \frac{1}{p-1}$$

and hence surely for $\vec{x} \in S$. It only remains to verify (6.6) for $\vec{x} = \vec{a} = p\vec{b} - \vec{\mu}$ with $a, b, \vec{\mu}$ satisfying (1.1), (1.4) and (6.3). This follows from Theorem 4 using (3.1.7) to deduce

$$(6.9) \quad \begin{aligned} & \gamma(M_m(\vec{a}), M_m(\vec{b}); \theta_m(\lambda), \theta_m(\lambda^p)) \\ &= \gamma(M_m(\vec{a}), M_m(\vec{b}); \theta_m(\lambda)) C_{M(\vec{b})}(\theta_m(\lambda^p), \theta_m(\lambda)^p)^t \end{aligned}$$

and using (6.4) to deduce

$$(6.10) \quad \begin{aligned} & \gamma(M_m(\vec{a}), M_m(\vec{b}), \theta_m(\lambda)) B(M_m(\vec{b}), M_m(\vec{b}) + \hat{\mu}_m, \theta_m(\lambda)^p)^{-1} \\ &= \gamma(M_m(\vec{a}), M_m(\vec{b}) + \hat{\mu}_m, \theta_m(\lambda)) = \Gamma_B(M_m(\vec{a}), \theta_m(\lambda)), \end{aligned}$$

the point being that (6.5) implies

$$(6.11) \quad p(M_m(\vec{b}) + \hat{\mu}_m) = M_m(\vec{a}) + \text{Rep} - M_m(\vec{a}).$$

This completes the proof of the theorem.

6.12. COROLLARY. $\Gamma_B(\vec{x}, \lambda)$ is analytic in \vec{x} and λ for x in S and $\lambda \neq \infty$;

$$\text{ord}(\lambda - 1) < \frac{1}{(p-1)} \left[1 - \frac{1}{(p-1)} \left(2 - \frac{1}{p-1} - \frac{1}{p} \right) \right].$$

PROOF. This has been explained (subsection 1.10) for $\text{ord}(\lambda - 1) \leq 0$. It only remains to remove this constraint. Using the theorem with $m = 5$, we obtain

$$(6.12.1) \quad \begin{aligned} & (-1)^{\mu_2} \Gamma_B(x_1, x_2, x_1 + x_2 - x_3; (1 - \lambda)) \\ &= \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} \Gamma_B(\vec{x}, \lambda) \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix} C_{(z_1, z_2, z_1 + z_2 - 3)}((1 - \lambda)^p, 1 - \lambda^p)^t \\ & \cdot \begin{cases} B(z_1, z_2, z_1 + z_2 - z_3; z_1, z_2, z_1 + z_2 - z_3 - 1; (1 - \lambda)^p)^{-1} & \text{if } \mu_1 + \mu_2 \geq \mu_3 + p, p + \mu_3 > \mu_1 + \mu_2 \geq \mu_3, \\ I_2 \\ B(z_1, z_2, z_1 + z_2 - z_3; z_1, z_2, z_1 + z_2 - z_3 + 1; (1 - \lambda)^p)^{-1} & \text{if } \mu_3 > \mu_1 + \mu_2 \end{cases} \end{aligned}$$

valid for $\vec{x} \in S$ and λ satisfying (1.9.1). The fact of $\Gamma_B(\vec{x}, \lambda)$ on the right may be extended to cover the disk $D(0, 1^-)$ except possibly for a pole at $\lambda = 0$. Furthermore, the factor $C_{(z_1, z_2, z_1 + z_2 - z_3)}((1 - \lambda)^p, 1 - \lambda^p)$ is by Lemma 1.13 analytic as a function of \vec{z}, λ in the region

$$(6.12.2) \quad \text{Inf}(0, \text{ord } \vec{z}) - \text{ord } \lambda^p + \text{ord}((1 - \lambda) - (1 - \lambda)^p) > \frac{1}{p-1}.$$

(In this argument we may assume $|\lambda| < 1$.) Since $\text{ord } \vec{z} > -e_2$ and

$$\text{ord}((1 - \lambda^p) - (1 - \lambda)^p) \geq 1 + \text{ord } \lambda,$$

condition (6.12.2) becomes

$$1 - e_2 - (p-1) \text{ord } \lambda > \frac{1}{p-1}.$$

The corollary now follows, remembering to replace λ by $1 - \lambda$, and observing that the possible singular locus introduced by the matrix B in (6.12.1) may be ruled out (for example) by comparison with (6.6) with $m = 3$.

6.13. COROLLARY. *The transformation formula (6.6) remains valid for \vec{x} in S and condition (1.9.1) replaced by*

$$(6.13.1) \quad \inf(|\lambda|, |1 - \lambda|, |1/\lambda|) > 1 - \epsilon$$

for some real number $\epsilon > 0$.

PROOF. The assertion follows from analytic continuation and Corollary 6.12.

6.14. COROLLARY. *For $x \in \bar{S}$ and λ satisfying (6.13.1),*

$$\Gamma_B((x_2, x_1, x_3), \lambda) \zeta(\vec{x}) = \begin{pmatrix} x_3 - x_1, & 0 \\ 0, & x_3 - x_2 \end{pmatrix} \Gamma_B(\vec{x}, \lambda) \begin{pmatrix} z_3 - z_1, & 0 \\ 0, & z_3 - z_2 \end{pmatrix}^{-1},$$

where

$$\zeta(\vec{x}) = (-1)^{\mu_2} \frac{\Gamma_p(1 + x_3 - x_2)}{\Gamma_p(1 - x_2) \Gamma_p(x_1) \Gamma_p(x_3 - x_1)} \cdot J_1(\vec{x}) J_2(\vec{x}),$$

$$J_2(\vec{x}) = \begin{cases} p, & \mu_3 < 1 + \mu_2, \\ \frac{1}{z_3 - z_2}, & 1 + \mu_2 \leq \mu_3, \end{cases} \quad J_1(\vec{x}) = \begin{cases} \frac{1}{p(z_3 - z_1)}, & \mu_3 < \mu_1, \\ 1, & \mu_1 \leq \mu_3. \end{cases}$$

PROOF. This follows directly from Theorem 5 and [4, equation (1.11)].

7. The Kummer group. Let us refer to the group of mappings $(a, b, c) \mapsto (\hat{a}, \hat{b}, \hat{c}) \bmod \mathbf{Z}$ in Table I as the Kummer group. Let us refer to the group of 48 elements generated by these together with the twist $(a, b, c) \mapsto (b, a, c)$, as the extended Kummer group. We give an intrinsic distinction between these two groups (viewed as subgroups of $\text{GL}(3, \mathbf{Z})$).

Let us view $0, a, b, c$ elements of \mathbf{R}/\mathbf{Z} , i.e. as points on the real circle. If $(a, b, c) \mapsto (\hat{a}, \hat{b}, \hat{c})$ denotes an element of the extended Kummer group then:

7.1. The points $0, c$ are separated by the points a, b (on the circle) if and only if the points $0, \hat{c}$ are separated by the points \hat{a}, \hat{b} .

7.2. If the points $0, c$ are separated by the points a, b then the orientation of $0, a, c, b$ coincides with that of $0, \hat{a}, \hat{c}, \hat{b}$ if $(a, b, c) \mapsto (\hat{a}, \hat{b}, c)$ lies in the Kummer group but not if it lies in the other coset of the Kummer group in the extended group.

7.3. REMARK. If the points $0, c$ are not separated by a, b then the orientation of $\hat{a}\hat{b}\hat{c}$ and abc need not coincide for the Kummer group.

7.4. We may check properties 7.1 (resp. 7.2) for the extended (resp. nonextended) Kummer group by checking the generators M_{13}, M_7 of the Kummer group and the twist for the extended Kummer group.

A second verification of 7.1 for the extended Kummer group comes from the condition for the existence of two solutions mod p of the hypergeometric differential equation.

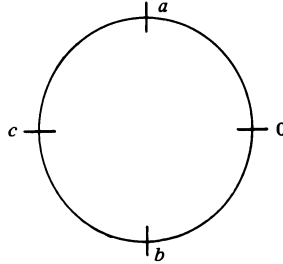
A third verification of 7.1 and 7.2 involves the dependence of the magnitude of $\det \gamma(\vec{a}, \vec{b}, \lambda)$ (λ satisfying (1.9.1)) upon the relative magnitudes of μ_1, μ_2, μ_3 (the representatives in $[0, p-1]$ of $-a_1, -a_2, -a_3$).

7.5. LEMMA. *The extended Kummer group as a subgroup of $GL(3, \mathbf{Z})$ is characterized by property 7.1. The Kummer group is characterized by the further property 7.2.*

PROOF. It is enough to show that there are at most 48 elements of $GL(3, \mathbf{Z})$

$$(7.5.1) \quad (a, b, c) \mapsto (\hat{a}, \hat{b}, \hat{c}) = (a, b, c) \begin{pmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{pmatrix}$$

which satisfy 7.1.



Consider a, b separated by $0, c$ as indicated. Then \hat{a}, \hat{b} (as given by (7.5.1)) separate $0, \hat{c}$. Let c, b be fixed and permit a to approach 0 . When a passes 0 , the points \hat{a}, \hat{b} can no longer separate $0, \hat{c}$. Hence when $a = 0$ the diagram $\hat{a}, \hat{b}, \hat{c}, 0$ must collapse. We assume here that b and c are *generic*. The collapse can occur in four ways,

$$(7.5.2) \quad \begin{array}{ll} (1) & \hat{a} = 0, \quad \text{i.e.} \quad bl_2 + cl_3 = 0, \\ (2) & \hat{a} = \hat{c}, \quad bl_2 + cl_3 = bn_2 + cn_3, \\ (3) & \hat{b} = 0, \quad bm_2 + cm_3 = 0, \\ (4) & \hat{b} = \hat{c}, \quad bm_2 + cm_3 = bn_2 + cn_3. \end{array}$$

Now clearly (1) and (2) cannot occur simultaneously and neither can (3) and (4). We assert that indeed no two can occur simultaneously. If (1) and (3) occur when $a = 0$ then (since b, c are generic) $l_2 = l_3 = 0, m_2 = m_3 = 0$ which shows that the transformation is not invertible, a contradiction. A similar analysis rules out the remaining $\binom{4}{2}$ -3 pairs. In this way we deduce a *mapping* depending upon (7.5.1) of the set of 4 situations

$$(7.5.3) \quad \begin{array}{ll} (1) & a = 0, \\ (2) & a = c, \\ (3) & b = 0, \\ (4) & b = c \end{array}$$

into the set of 4 situations (7.5.2) in terms of $\hat{a}, \hat{b}, \hat{c}$. By using the inverse of (7.5.1) we conclude that there exists a mapping of the situations (7.5.2) into the situations

(7.5.3), i.e. the mapping (7.5.1) satisfying 7.1 determines a one-to-one mapping of the situations (7.5.2) onto the situations (7.5.3). There are exactly $4 \cdot 3 \cdot 2 \cdot 1 = 24$ such mappings. On the other hand, a one-to-one correspondence between (7.5.2) and (7.5.3) determines the matrix of (7.5.1) up to a scalar factor. For example, suppose

$$\begin{aligned} a = 0 &\Rightarrow \hat{a} = 0, \\ a = c &\Rightarrow \hat{a} = \hat{c}, \\ b = 0 &\Rightarrow \hat{b} = 0, \\ b = c &\Rightarrow \hat{b} = \hat{c}, \end{aligned}$$

then

$$\begin{aligned} l_2 = l_3 = 0, \quad l_2 = n_2, \quad l_1 + l_3 = n_1 + n_3, \\ m_1 = m_3 = 0, \quad m_1 = n_1, \quad m_2 + m_3 = n_2 + n_3, \end{aligned}$$

from which we deduce that the matrix of (7.5.1) is a scalar, $l_1 = m_2 = n_3$.

Since the matrix (7.5.1) has determinant ± 1 , there are at most two choices of (7.5.1) for each 1-1 correspondence between (7.5.2) and (7.5.3). This completes the proof.

REFERENCES

1. A. Adolphson and S. Sperber, *Twisted Kloosterman sums and p-adic Bessel functions*, Amer. J. Math. (to appear).
2. W. Burnside, *Theory of groups of finite order*, Dover, New York, 1955.
3. B. Dwork, *Lectures on p-adic differential equations*, Springer-Verlag, Berlin and New York, 1982.
4. ———, *On the Boyarsky principle*, Amer. J. Math. **105** (1983), 115–156.
5. A. Erdelyi, *Integral darstellungen hypergeometrischen functionen*, Quart. J. Math. Oxford Ser. **8** (1937), 267–277.
6. ———, *Transformation of hypergeometric integrals by means of fractional integration by parts*, Quart. J. Math. Oxford Ser. **10** (1939), 176–189.
7. A. Erdelyi, W. Magnus and F. G. Tricomi, *Higher transcendental functions*, Vol. I, McGraw-Hill, New York, 1953.
8. P. Monsky and G. Washnitzer, *Formal cohomology*. I, Ann. of Math. (2) **88** (1968), 181–217.
9. P. Monsky, *Formal cohomology*. II, Ann. of Math. (2) **88** (1968), 218–238.
10. ———, *Formal cohomology*. III, Ann. of Math. (2) **93** (1971), 315–343.
11. E. Poole, *Introduction to the theory of linear differential equations*, Oxford Univ. Press, Oxford, 1936.
12. B. Riemann, *Gesammelte Math Werke*, Dover, New York, 1953.

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